

# Relativistic analysis of magnetoelectric crystals: extracting a new 4-dimensional $P$ odd and $T$ odd pseudoscalar from $\text{Cr}_2\text{O}_3$ data

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Earlier, the linear magnetoelectric effect of chromium sesquioxide  $\text{Cr}_2\text{O}_3$  has been determined experimentally as a function of temperature. One measures the electric field-induced magnetization on  $\text{Cr}_2\text{O}_3$  crystals or the magnetic field-induced polarization. From the magnetoelectric moduli of  $\text{Cr}_2\text{O}_3$  we extract a 4-dimensional relativistic invariant pseudoscalar  $\tilde{\alpha}$ . It is temperature dependent and of the order of  $\sim 10^{-4} Y_0$ , with  $Y_0$  as vacuum admittance. We show that the new pseudoscalar  $\tilde{\alpha}$  is odd under parity transformation and odd under time inversion. Moreover,  $\tilde{\alpha}$  is for  $\text{Cr}_2\text{O}_3$  what Tellegen's *gyrator* is for two port theory, the *axion* field for axion electrodynamics, and the PEMC (perfect electromagnetic conductor) for electrical engineering.

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## I. INTRODUCTION

Our paper addresses the magnetoelectric (ME) effect. This effect has been established since the 1960's in  $\text{Cr}_2\text{O}_3$  crystals (see the reviews by O'Dell [1] and, more recently, by Fiebig [2]). The ME effect, in linear approximation, is described by magnetoelectric susceptibilities or moduli that have been measured by different groups.

It has been a long-standing discussion whether these moduli fulfill a certain condition, as predicted by Post in 1962 [3]. This condition was dubbed *Post constraint* by Lakhtakia, see, e.g., [4],[5]. Numerous arguments against the validity of the Post constraint were put forward, some of them are mentioned in [6] and [7], e.g.. However, in the end, we must turn to the experiments and their proper evaluation.

Following Post [3], we provide here a relativistic invariant formalism of the electrodynamics of moving media. The violation of the Post constraint is measured by a *pseudoscalar* modulus, which has not been determined so far. If this pseudoscalar vanishes, the Post constraint is fulfilled, otherwise it is violated. On the basis of experimental data, we determine this pseudoscalar (or axion) piece of the magnetoelectric moduli and find it *non-vanishing*. Therefore the Post constraint cannot be

uphold as a general valid relation.

However, our paper has also an interdisciplinary purpose. Our result of the non-vanishing pseudoscalar provides a physical structure that also shows up in the theory of electric networks, more exactly in the theory of two ports, as Tellegen's gyrator [8], in electrical engineering as perfect electromagnetic conductor (PEMC) [9], and in elementary particle physics as the hypothetical axion field [10],[11]. These interrelationships support each other. Since the axion in elementary physics is the only left hypothetical object in this context, our results make also the existence of the axion particle more likely.

In Sec.2, we give a short description of the ME effect. In Sec.3, a 4-dimensional electrodynamic framework for moving media is built up and the electromagnetic constitutive tensor introduced for local and linear media. In Sec.4 we discuss Dzyaloshinskii's theory [12] of  $\text{Cr}_2\text{O}_3$  and extract therefrom the mentioned pseudoscalar. In Sec.5 we finally determine the pseudoscalar for the first time and discuss some of its properties in Sec.6. Then, in Sec.7, we turn to the interdisciplinary part and discuss the existence of the pseudoscalar for network theory, electrical engineering, and elementary particle physics. In the concluding section, we collect our results.

## II. MAGNETOELECTRIC EFFECT

In classical electrodynamics for a local linear medium, which is at rest in the reference frame considered, the constitutive law reads  $\mathbf{D} = \epsilon\epsilon_0\mathbf{E}$  and  $\mathbf{H} = \mathbf{B}/(\mu\mu_0)$ . Here  $\epsilon_0$  is the electric constant (permittivity of free space) and  $\mu_0$  the magnetic constant (permeability of free space), whereas  $\epsilon$  and  $\mu$  are the (relative) permittivity and permeability, respectively, of the medium under consider-

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ation. Furthermore, the admittance of free space is  $Y_0 = 1/\Omega_0 = \sqrt{\varepsilon_0/\mu_0}$ , with  $\Omega_0$  as vacuum impedance, and the speed of light  $c = 1/\sqrt{\varepsilon_0\mu_0}$ .

If an external  $\mathbf{B}$  field in some suitable medium induces an electric excitation  $\mathbf{D}$  and an external  $\mathbf{E}$  field a magnetic excitation  $\mathbf{H}$ , the constitutive law mentioned has to be extended by so-called magnetoelectric pieces, see O'Dell [1]. The general *local* and *linear* constitutive law, if the medium is anisotropic, reads

$$D = (\varepsilon)\varepsilon_0 E + (\alpha_1)Y_0 B, \quad (1)$$

$$H = (\alpha_2)Y_0 E + (\mu^{-1})\mu_0^{-1} B. \quad (2)$$

We have to read (1) and (2) as tensor equations, with  $(\varepsilon)$ ,  $(\mu^{-1})$ ,  $(\alpha_1)$ , and  $(\alpha_2)$  as dimensionless  $3 \times 3$  matrices. Hence we expect 36 permittivity, permeability, and magnetoelectric moduli in general. The constants  $\varepsilon_0$ ,  $Y_0$ , and  $\mu_0$  are required for dimensional consistency.

The existence of nonvanishing  $(\alpha_1)$  and  $(\alpha_2)$  matrices was foreseen by Landau-Lifshitz [13] for certain magnetic crystals and proposed by Dzyaloshinskii [12] specifically for the antiferromagnet  $\text{Cr}_2\text{O}_3$ . Astrov [14] (for an electric field) and Rado & Folen [15] (for a magnetic field) confirmed this theory experimentally for  $\text{Cr}_2\text{O}_3$  crystals. For reviews, see [16] — there other magnetoelectric crystals are listed, too — and [2].

### III. FOUR-DIMENSIONAL MAXWELLIAN FRAMEWORK

In order to extend the formalism to systems *moving* in the reference frame considered, but also in order to recognize the relativistic covariant structures of (1) and (2), we have to go over to a four-dimensional (4D) formalism. We collect  $D$  and  $H$  in the 4D excitation tensor density  $\mathfrak{G}^{\mu\nu}(D, H) = -\mathfrak{G}^{\nu\mu}$  and  $E$  and  $B$  in the 4D field strength tensor  $F_{\mu\nu}(E, B) = -F_{\nu\mu}$ , with  $\mu, \nu, \dots = 0, 1, 2, 3$  and coordinates  $(x^0 = t, x^1, x^2, x^3)$ , see Post [3]. Then the Maxwell equations read

$$\partial_\nu \mathfrak{G}^{\mu\nu} = \mathfrak{J}^\mu, \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0, \quad (3)$$

with  $\mathfrak{J}^\mu(\rho, j)$  as 4-current. This is the “premetric” form of Maxwell’s equations. They are covariant under general coordinate transformations and do not depend on the metric of spacetime, that is, they are valid in this form in special and in general relativity alike.

In order to complete the Maxwell equations (3) to a predictive physical system, we have to specify a constitutive law linking  $\mathfrak{G}^{\mu\nu}$  to  $F_{\mu\nu}$ . In vacuum, here now eventually the metric  $g_{\mu\nu}$  of spacetime enters with signature  $(-+++)$ , we have

$$\mathfrak{G}^{\lambda\nu} = Y_0 \sqrt{-g} g^{\lambda\alpha} g^{\nu\beta} F_{\alpha\beta} = Y_0 \sqrt{-g} F^{\lambda\nu}, \quad (4)$$

with  $g := \det g_{\rho\sigma} \neq 0$ . From the covariant components of the metric  $g_{\mu\nu}$ , its contravariant components  $g^{\lambda\nu}$  can be determined via  $g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu$ .

For magnetoelectric media that are local and linear, we assume, following Tamm [17] and Post [3], the constitutive law

$$\mathfrak{G}^{\lambda\nu} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa}, \quad (5)$$

where  $\chi^{\lambda\nu\sigma\kappa}$  is a *constitutive tensor density* of rank 4 and weight +1, with the dimension  $[\chi] = 1/\text{resistance}$ . Since both  $\mathfrak{G}^{\lambda\nu}$  and  $F_{\sigma\kappa}$  are antisymmetric in their indices, we have  $\chi^{\lambda\nu\sigma\kappa} = -\chi^{\lambda\nu\kappa\sigma} = -\chi^{\nu\lambda\sigma\kappa}$ . An antisymmetric pair of indices corresponds, in 4D, to six independent components. Thus, the constitutive tensor can be considered as a  $6 \times 6$  matrix with 36 independent components, see (1) and (2).

A  $6 \times 6$  matrix can be decomposed in its tracefree symmetric part (20 independent components), its antisymmetric part (15 components), and its trace (1 component). On the level of  $\chi^{\lambda\nu\sigma\kappa}$ , this *decomposition* is reflected in [18]

$$\chi^{\lambda\nu\sigma\kappa} = {}^{(1)}\chi^{\lambda\nu\sigma\kappa} + {}^{(2)}\chi^{\lambda\nu\sigma\kappa} + {}^{(3)}\chi^{\lambda\nu\sigma\kappa}. \quad (6)$$

$$36 = 20 \oplus 15 \oplus 1.$$

The third part, the *axion* part, is totally antisymmetric and as such proportional to the Levi-Civita symbol,  ${}^{(3)}\chi^{\lambda\nu\sigma\kappa} := \chi^{[\lambda\nu\sigma\kappa]} = \tilde{\alpha} \tilde{\epsilon}^{\lambda\nu\sigma\kappa}$ . Here, the totally antisymmetric Levi-Civita symbol is  $\tilde{\epsilon}_{\lambda\nu\sigma\kappa} = \pm 1, 0$ ; we denote pseudotensors with a tilde. The second part, the *skewon* part, is defined according to  ${}^{(2)}\chi^{\mu\nu\lambda\rho} := \frac{1}{2}(\chi^{\mu\nu\lambda\rho} - \chi^{\lambda\rho\mu\nu})$ . If the constitutive equation can be derived from a Lagrangian, which is the case as long as only reversible processes are considered, then  ${}^{(2)}\chi^{\lambda\nu\sigma\kappa} = 0$ . We will assume this condition henceforth. Below, for  $\text{Cr}_2\text{O}_3$ , it will be verified experimentally. The *principal* part  ${}^{(1)}\chi^{\lambda\nu\sigma\kappa}$  has the symmetries  ${}^{(1)}\chi^{\lambda\nu\sigma\kappa} = {}^{(1)}\chi^{\sigma\kappa\lambda\nu}$  and  ${}^{(1)}\chi^{[\lambda\nu\sigma\kappa]} = 0$ . The tensor  $\chi^{\lambda\nu\sigma\kappa}$  has now  $20 + 1$  independent components and the constitutive law reads

$$\mathfrak{G}^{\lambda\nu} = \frac{1}{2} \left( {}^{(1)}\chi^{\lambda\nu\sigma\kappa} + \tilde{\alpha} \tilde{\epsilon}^{\lambda\nu\sigma\kappa} \right) F_{\sigma\kappa}. \quad (7)$$

We can express the axion piece  $\tilde{\alpha}$  directly in the constitutive tensor (6). With  ${}^{(2)}\chi^{\lambda\nu\sigma\kappa} = 0$ , we find

$$\tilde{\alpha} = \frac{1}{4!} \tilde{\epsilon}_{\lambda\nu\sigma\kappa} \chi^{\lambda\nu\sigma\kappa} = \frac{1}{3} (\chi^{0123} + \chi^{0231} + \chi^{0312}). \quad (8)$$

This 4D pseudoscalar (or axion piece) of  $\chi^{\lambda\nu\sigma\kappa}$  will be determined for  $\text{Cr}_2\text{O}_3$ .

We split (7) into space and time [18]. Then we recover equations of the form of (1) and (2), but with an exact relativistic meaning ( $a, b, \dots = 1, 2, 3$ ):

$$D^a = \varepsilon^{ab} E_b + \gamma^a_b B^b + \tilde{\alpha} B^a, \quad (9)$$

$$H_a = \mu_{ab}^{-1} B^b - \gamma^b_a E_b - \tilde{\alpha} E_a. \quad (10)$$

We have the 6 permittivities  $\varepsilon^{ab} = \varepsilon^{ba}$ , the 6 permeabilities  $\mu_{ab} = \mu_{ba}$ , and the 8+1 magnetoelectric pieces  $\gamma^a_b$  (its trace vanishes,  $\gamma^c_c = 0$ ) and  $\tilde{\alpha}$ , respectively. Equivalent constitutive relations were formulated by Serdyukov et al. [19], p.86, and studied in quite some detail. It is remarkable, as can be recognized from (7) and (8), that  $\tilde{\alpha}$  is a 4D pseudoscalar.

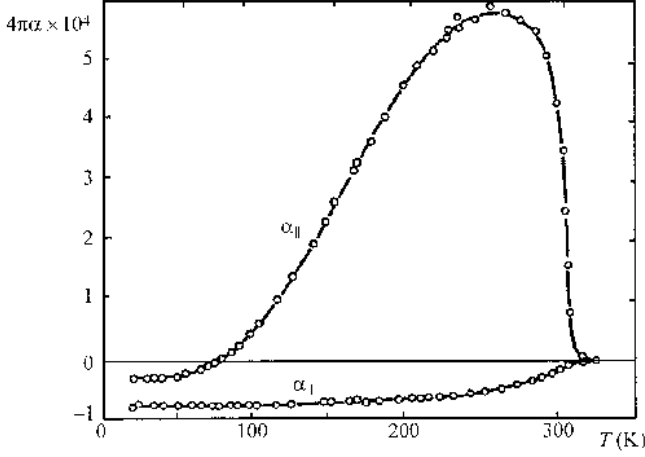


FIG. 1: The ME<sub>E</sub> effect (linear magnetoelectric effect with electric field-induced magnetization) of Cr<sub>2</sub>O<sub>3</sub>: Temperature dependence of the magnetoelectric components  $\alpha_{||}$  and  $\alpha_{\perp}$  according to Astrov [14].

#### IV. CHROMIUM OXIDE Cr<sub>2</sub>O<sub>3</sub> AND ITS CONSTITUTIVE LAW

On the basis of neutron scattering data and susceptibility measurements of the antiferromagnetic chromium sesquioxide Cr<sub>2</sub>O<sub>3</sub>, Dzyaloshinskii [12] was able to establish the magnetic symmetry class  $\bar{3}'m'$  of a Cr<sub>2</sub>O<sub>3</sub> crystal. On this basis, he formulated the following constitutive law for Cr<sub>2</sub>O<sub>3</sub>:

$$D_{x,y} = \varepsilon_{\perp} \varepsilon_0 E_{x,y} + \frac{\alpha_{\perp}}{c} H_{x,y}, \quad (11)$$

$$D_z = \varepsilon_{||} \varepsilon_0 E_z + \frac{\alpha_{||}}{c} H_z, \quad (12)$$

$$B_{x,y} = \mu_{\perp} \mu_0 H_{x,y} + \frac{\alpha_{\perp}}{c} E_{x,y}, \quad (13)$$

$$B_z = \mu_{||} \mu_0 H_z + \frac{\alpha_{||}}{c} E_z. \quad (14)$$

The  $z$ -axis is parallel to the trigonal (and the optical) axis of the crystal. The permittivities parallel and perpendicular to the  $z$ -axis are denoted by  $\varepsilon_{||}, \varepsilon_{\perp}$ , analogously the permeabilities by  $\mu_{||}, \mu_{\perp}$ , and the magnetoelectric moduli by  $\alpha_{||}, \alpha_{\perp}$ . Note that all these moduli are dimensionless (in *all* systems of units).

The theory (11) to (14) and also the corresponding measurements were made in the  $(D, B)$  system. However, in order to get to the manifestly relativistic covariant  $(D, H)$  representation (9),(10), we have to resolve the  $(D, B)$  system with respect to  $D$  and  $H$  and to compare the corresponding coefficients. We find magnetoelectric matrix

$$\gamma^a_b = \frac{Y_0}{3} \left( \frac{\alpha_{\perp}}{\mu_{\perp}} - \frac{\alpha_{||}}{\mu_{||}} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (15)$$

and the pseudoscalar or axion piece

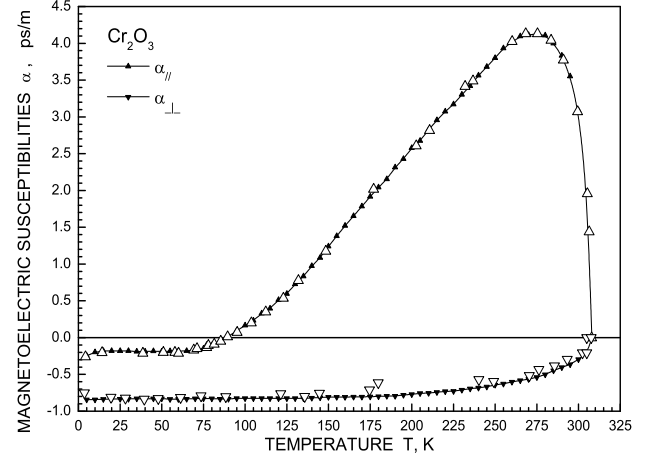


FIG. 2: The ME<sub>H</sub> effect (linear magnetoelectric effect with magnetic field-induced polarization) of Cr<sub>2</sub>O<sub>3</sub>: Temperature dependence of the magnetoelectric moduli  $\alpha_{||}$  and  $\alpha_{\perp}$  according to Wiegelmann et al. [20], Fig.2. Their measured values are denoted by large empty triangles, pointing up or down; superimposed are the interpolated and digitalized point, small black triangles, pointing up or down. The curve was determined by using B-splines. One should also compare Rivera [21] and Wiegelmann [22]. The relative sign between  $\alpha_{||}$  and  $\alpha_{\perp}$  was taken from Astrov [14]. The magnetic field  $\mu_0 H$  was below 6 tesla.

$$\tilde{\alpha} = \frac{Y_0}{3} \left( 2 \frac{\alpha_{\perp}}{\mu_{\perp}} + \frac{\alpha_{||}}{\mu_{||}} \right). \quad (16)$$

Eqs.(15) and (16) can be collected in the “relativistic”  $\alpha$ -matrix

$${}^{\text{rel}}\alpha^a_b := \gamma^a_b + \tilde{\alpha} \delta^a_b = Y_0 \begin{pmatrix} \frac{\alpha_{\perp}}{\mu_{\perp}} & 0 & 0 \\ 0 & \frac{\alpha_{\perp}}{\mu_{\perp}} & 0 \\ 0 & 0 & \frac{\alpha_{||}}{\mu_{||}} \end{pmatrix}. \quad (17)$$

#### V. MEASUREMENTS ON Cr<sub>2</sub>O<sub>3</sub>

Astrov [14], see Fig.1, measured  $\alpha_{||}$  and  $\alpha_{\perp}$  in an *electric* field  $E$  according to (13) and (14) and Rado & Folen in a *magnetic* excitation  $H$  according to (11) and (12). Within the measurement limits, they found the same values. This verifies that  ${}^{(2)}\chi^{\lambda\nu\sigma\kappa} = 0$  and confirms Dzyaloshinskii’s theory (below the spin-flop phase). Later on, mostly measurements in magnetic fields below  $\mu_0 H = 6 T$  were made, see Rivera [21] and Wiegelmann et al. [20].

Measurements of Wiegelmann et al. [20] are plotted in Fig.2. The maximum of  $\alpha_{||}$  was found at about 275 K:

$$\alpha_{||} (\text{at } 275 K) \approx 4.13 \frac{\text{ps}}{m} \times c \approx 1.238 \times 10^{-3}. \quad (18)$$

Now we can compute from the values of Fig.2 the pseudoscalar (16). However, we need additionally the permeabilities  $\mu_{||}$  and  $\mu_{\perp}$ . They have been measured by Foner

[23]. We find at 4.2 K,  $\mu_{\perp} \approx 1.00147$  and the maximum value near the Néel temperature  $\approx 1.00162$ . Since  $\mu_{\parallel}$  deviates even less from 1, we have  $\mu \approx 1$ . Then (16) can be easily evaluated:  $\tilde{\alpha} \approx \frac{1}{3} (2\alpha_{\perp} + \alpha_{\parallel}) Y_0$ . Our results are plotted in Fig.3.

As we can see, for temperatures of up to about 163 K, the pseudoscalar is negative, for higher temperatures positive until it vanishes at the Néel temperature of about 308 K. For the maximum, we find

$$\tilde{\alpha}_{\max} (\text{at } 285 \text{ K}) \approx 3.10 \times 10^{-4} Y_0 \stackrel{\text{SI}}{\approx} 0.822 \frac{1}{M\Omega}. \quad (19)$$

We conclude that  $\tilde{\alpha}$  is fairly small but, for  $T \neq 163 \text{ K}$ , definitely nonvanishing.

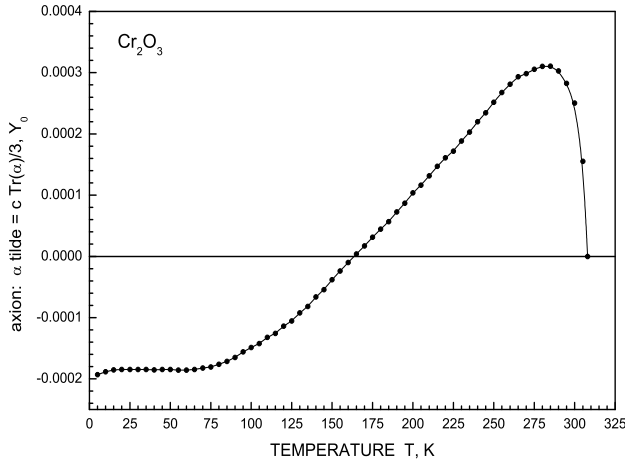


FIG. 3: Our new result: The pseudoscalar or axion piece  $\tilde{\alpha}$  of the constitutive tensor  $\chi^{\lambda\nu\sigma\kappa}$  of  $\text{Cr}_2\text{O}_3$  in units of  $Y_0$  as a function of the temperature  $T$  in *kelvin*; here  $Y_0$  is the vacuum admittance, which is 1 in Gaussian units and about  $1/(377 \text{ ohm})$  in SI. In the figure the tilde of  $\tilde{\alpha}$  is missing.

Does this result imply consequences also for the experimentalist? We think so for the following reason: As we saw above,  $\mu_{\parallel}$  as well as  $\mu_{\perp}$  are approximately one. Therefore the magnetoelectric  $\gamma$  matrix (15) becomes

$$\gamma^a_b \approx \frac{Y_0}{3} (\alpha_{\perp} - \alpha_{\parallel}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (20)$$

The question is now: *Can we find a substance in which*

$$\alpha_{\perp} = \alpha_{\parallel}, \quad (21)$$

that is, in which the matrix  $\gamma^a_b$  vanishes *for all temperatures*? This challenge for experimentalists would be interesting in the sense that then one would have a substance in which the *only* magnetoelectric piece would be the pseudoscalar (or axion) piece  $\tilde{\alpha}$ . In other words, this substance would display an *isotropic magnetoelectric effect*.

For a theoretician it could be of value if he/she looks for a microscopic Hamiltonian. The pseudoscalar  $\tilde{\alpha}$  of the magnetoelectric effect should have a different physical origin as compared to the  $\gamma$  part. Thus, it could be helpful for developing microscopic models for the magnetoelectric effect.

## VI. PROPERTIES OF THE PSEUDOSCALAR OR AXION PIECE

Unlike the 3D vectors of the electric and the magnetic fields and the 3D tensors of permittivity  $\epsilon^{ab}$ , of permeability  $\mu^{ab}$ , and of the magnetoelectric moduli  $\gamma^a_b$ , which all depend on the choice of the reference frame and the local coordinates, the value of  $\tilde{\alpha}$  is always the *same*. It is invariant under any orientation-preserving transformation of frames and coordinates — and changes sign when the orientation is changed.

If we consider the pseudoscalar  $\tilde{\alpha}$  alone, then we can take its constitutive law from (9) and (10),

$$D^a = +\tilde{\alpha} B^a, \quad H_a = -\tilde{\alpha} E_a \quad (22)$$

or, in 4D,

$$\mathfrak{G}^{\lambda\nu} = \tilde{\alpha} \tilde{\epsilon}^{\lambda\nu\sigma\kappa} F_{\sigma\kappa} / 2. \quad (23)$$

A space reflection

$$D^a \rightarrow -D^a, H_a \rightarrow H_a, E_a \rightarrow -E_a, B^a \rightarrow B^a, \quad (24)$$

as well and a time inversion

$$D^a \rightarrow D^a, H_a \rightarrow -H_a, E_a \rightarrow E_a, B^a \rightarrow -B^a, \quad (25)$$

see Janner [24] and Marmo et al. [25], will turn (22) into its negative,

$$D^a = -\tilde{\alpha} B^a, \quad H_a = +\tilde{\alpha} E_a. \quad (26)$$

This is an expression of the *pseudoscalar* nature of  $\tilde{\alpha}$ . Therefore  $\tilde{\alpha}$  is P odd and T odd.

Moreover, the energy-momentum tensor for the electromagnetic field, see [3, 18],

$$\mathfrak{T}_{\lambda}^{\nu} = \frac{1}{4} \mathfrak{G}^{\sigma\tau} F_{\sigma\tau} \delta_{\lambda}^{\nu} - \mathfrak{G}^{\nu\sigma} F_{\lambda\sigma}, \quad (27)$$

if (23) is substituted, vanishes:

$$\mathfrak{T}_{\lambda}^{\nu} (\text{of axion piece } \tilde{\alpha}) = 0. \quad (28)$$

Thus, the electromagnetic energy density  $\mathfrak{T}_0^0$ , the energy-flux density (Poynting flux)  $\mathfrak{T}_0^b$  etc. of the axion piece vanish.

## VII. ANALOGUES OF THE 4D PSEUDOSCALAR $\tilde{\alpha}$ IN NETWORK THEORY, IN ELECTRICAL ENGINEERING, AND IN PARTICLE PHYSICS

The structure of the constitutive law (22) or (23) is not unprecedented. In electrical engineering, in the theory linear networks, more specifically in the theory of two ports (or four poles), Tellegen [8, 26] came up with the new structure of a *gyrator*, which is defined via

$$v_1 = -s i_2, \quad v_2 = s i_1, \quad (29)$$

where  $v$  are voltages and  $i$  currents of the ports 1 and 2, respectively. Let us quote from Tellegen [26], p.189: “The ideal gyrator has the property of ‘gyrating’ a current into a voltage, and vice versa. The coefficient  $s$ , which has the dimension of a resistance, we call the gyration resistance;  $1/s$  we call the gyration conductance.” The gyrator is a nonreciprocal network element.

If we turn to the electromagnetic field, then because of dimensional reasons the quantities related to the *currents*  $i_1, i_2$  are the excitations  $D^a, H_a$  and the quantities related to the *voltages*  $v_1, v_2$  the field strengths  $E_a, B^a$ . Then we find straightforwardly the analogous relations

$$E_a = -s H_a, \quad B^a = s D^a. \quad (30)$$

If we rename the resistance  $s$  according to  $s = 1/\tilde{\alpha}$ , then (30) and (22) coincide. Without the least doubt, the gyrator is in the theory of electrical networks what the axion piece is in magnetoelectricity. The axion piece ‘rotates’ the excitations, modulo an admittance, into the field strengths, as the gyrator the currents into voltages.

These analogies or rather isomorphisms carry even further. In 2005, Lindell & Sihvola [9], see also [27], introduced the new concept of a *perfect electromagnetic conductor* (PEMC). It obeys the constitutive law (22) or (23). The PEMC is a generalization of the perfect electric and the perfect magnetic conductor. In this sense, it is the ‘ideal’ electromagnetic conductor that can be hopefully built by means of a suitable *metamaterial*, see Sihvola [28].

Continuing with our search for isomorphisms, we turn to axion electrodynamics, see Ni [10] and Wilczek [11] and, for more recent work, Itin [29], [30]. If for vacuum electrodynamics we add to the usual Maxwell-Lorentz

expression specified in (4) an axion piece patterned after the last term in (7), then we have the constitutive law for axion electrodynamics,

$$\mathfrak{G}^{\lambda\nu} = Y_0 \sqrt{-g} F^{\lambda\nu} + \frac{1}{2} \tilde{\alpha} \epsilon^{\lambda\nu\sigma\kappa} F_{\sigma\kappa}. \quad (31)$$

In  $\text{Cr}_2\text{O}_3$  we have  $\tilde{\alpha} \approx 10^{-4} Y_0$ . It is everybody’s guess what it could be for the physical vacuum. In elementary particle theory one adds in the corresponding Lagrangian also kinetic terms of the axion à la  $\sim g^{\mu\nu} \partial_\mu \tilde{\alpha} \partial_\nu \tilde{\alpha}$  and possibly a massive term  $\sim m_\alpha^2 \tilde{\alpha}^2$ . However, this hypothetical  $P$  odd and  $T$  odd particle has not been found so far, in spite of considerable experimental efforts, see Davis et al. [31].

The axion shares its  $P$  odd and  $T$  odd properties with the  $\tilde{\alpha}$  piece of  $\text{Cr}_2\text{O}_3$ , with the gyrator, and with the PEMC. One may speculate whether an axion detector made of  $\text{Cr}_2\text{O}_3$  crystals could enhance the probability of finding axions.

## VIII. CONCLUSIONS

Our results can be summed up as follows:

- The magnetoelectric pseudoscalar of  $\text{Cr}_2\text{O}_3$  is temperature dependent and of the order of  $10^{-4} Y_0$ . Thus, the Post constraint is invalid in general.
- Our result and the formalism is of general interest to areas such as multiferroics and similar materials where there is a need for a generalized and concise description of the phenomena.
- We suggest to the experimentalist to search for a substance with an *isotropic* magnetoelectric effect.
- Since the gyrator and the PEMC are established notions in network theory and in electrical engineering, these two notions, together with the isomorphic structure of the pseudoscalar piece in  $\text{Cr}_2\text{O}_3$ , support the existence of a fundamental axion field coupled to conventional vacuum electrodynamics. Thus, axion electrodynamics gains plausibility by our results.

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